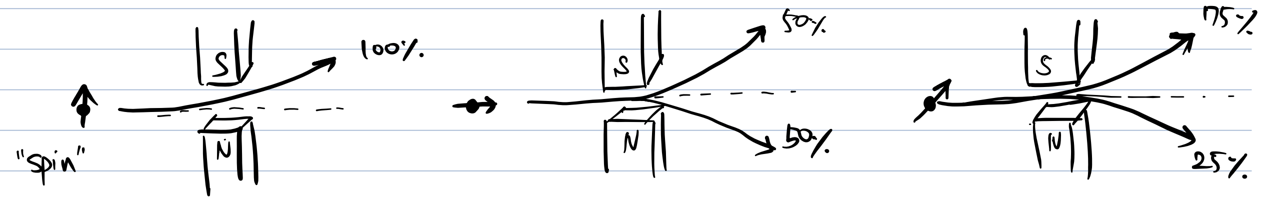
I. Bits and Qubits.

(1) Stern - Gerlach experiment.



"position is quantized."
(probabilities of up/down change)

Dirac notation

$\Rightarrow$ spin's state : $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ (Wave-function)

complex numbers

$$\begin{cases} P_\uparrow = |\alpha|^2 \\ P_\downarrow = |\beta|^2 \end{cases}$$: Born rule Prob.

(2) Mathematical formulation of quantum mechanics

- Hilbert space : $\mathcal{H}^d$ ← dimension (complex vector space with inner product)
- Basis : $\{|0\rangle, |1\rangle, \cdots, |d-1\rangle\}$ $\;,\; \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{d-1} \end{pmatrix}$, $\alpha_i \in \mathbb{C}$.
- Quantum state : $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle \in \mathcal{H}^d$ (vector)

  (bra) (ket) with $\langle\psi|\psi\rangle = 1$.
- Dual vector $\langle\psi| = \sum_{i=0}^{d-1} \alpha_i^* \langle i|$

  ← inner product
- Inner product between $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle$ & $|\phi\rangle = \sum_{i=0}^{d-1} \beta_i |i\rangle$

$$\langle\phi|\psi\rangle = \left(\sum_{j=0}^{d-1} \beta_j^* \langle j|\right)\left(\sum_{i=0}^{d-1} \alpha_i |i\rangle\right)$$

$$= \sum_{i,j=0}^{d-1} \beta_j^* \alpha_i \underbrace{\langle j|i\rangle}_{=\delta_{ij}} = \sum_{i=0}^{d-1} \alpha_i \beta_j^*$$

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$$

- $\langle\psi|\psi\rangle = 1 \Rightarrow \sum_{i=0}^{d-1} |\alpha_i|^2 = 1$. (normalization)

- Operator $\hat{O} : \mathcal{H}^d \mapsto \mathcal{H}^d$ (linear map from vector to vector)

$$\hat{O} = \sum_{i,j=0}^{d-1} O_{ij} |i\rangle\langle j| \Rightarrow \hat{O}|\psi\rangle = \sum_{i,j=0}^{d-1} O_{ij}\alpha_j |i\rangle$$

$$\hat{O}(|\phi\rangle + |\psi\rangle) = \hat{O}|\phi\rangle + \hat{O}|\psi\rangle \ \& \ \hat{O}(c|\psi\rangle) = c\hat{O}|\psi\rangle$$

(for $c \in \mathbb{C}$)

- Some special operators

1) Hermitian operator (a.k.a. Observables)

$$\hat{A} = \hat{A}^\dagger \quad \left(O^\dagger = \sum_{i,j=0}^{d-1} O_{ij}^* |j\rangle\langle i| = \sum_{i,j=0}^{d-1} O_{ji}^* |i\rangle\langle j| \right)$$

"conjugate transpose"

$\Rightarrow$ $\hat{A}$ has real eigenvalues.

$\Rightarrow$ $\langle \hat{A} \rangle_\psi = \langle\psi|\hat{A}|\psi\rangle \in \mathbb{R}, \ \forall |\psi\rangle \in \mathcal{H}^d$

$\hat{A} = \sum_i a_i \hat{P}_i$

: Expectation value of $\hat{A}$ (physically measureable)

(e.g.) $\hat{H}$: Hamiltonian $\quad E_\psi = \langle\psi|\hat{H}|\psi\rangle$.

2) Positive (semi-) definite operator

$$\hat{M} \geq 0 \iff \langle\psi|\hat{M}|\psi\rangle \geq 0, \ \forall |\psi\rangle \in \mathcal{H}^d$$

- Density operator ($\hat{\rho} \geq 0$ & $\mathrm{Tr}[\hat{\rho}] = 0$)

More general expression of quantum states.

3) Projection operator

$$\hat{P}^2 = \hat{P} \quad (\text{with } \hat{P} \geq 0)$$

(example) $\hat{P}_i = (|i\rangle\langle i|)^2 = (|i\rangle\langle i|)(|i\rangle\langle i|) = |i\rangle\langle i|$

$$\boxed{P_i = \langle\psi|\hat{P}_i|\psi\rangle \quad \text{(Born rule)}}$$

4) Identity operator

$$\hat{\mathbb{1}}|\psi\rangle = |\psi\rangle, \ \forall |\psi\rangle \in \mathcal{H}^d$$

$$\hat{\mathbb{1}} = \sum_{i=0}^{d-1} |i\rangle\langle i| \Rightarrow \sum_{i=0}^{d-1} |i\rangle\langle i| \left(\sum_{j=0}^{d-1} \alpha_j |j\rangle\right) = \sum_{i=0}^{d-1} \alpha_i |i\rangle$$

(completeness)

5) Unitary operator (inner product preserved)

$$\hat{U}^{-1} = \hat{U}^{\dagger} \iff \hat{U}^{\dagger}\hat{U} = \hat{\mathbb{1}} = \hat{U}\hat{U}^{\dagger}$$

$$|\psi'\rangle = \hat{U}|\psi\rangle, \quad |\phi'\rangle = \hat{U}|\phi\rangle$$

$$\Rightarrow \langle\phi'|\psi'\rangle = \langle\phi|\hat{U}^{\dagger}\hat{U}|\psi\rangle = \langle\phi|\psi\rangle \quad \forall\, |\phi\rangle, |\psi\rangle \in \mathcal{H}^d$$

(inner product is preserved)

Moreover,

$\langle\psi'|\psi'\rangle = \langle\psi|\psi\rangle = 1.$ $\therefore$ $|\psi'\rangle$ is also a quantum state.

(example)

$$(i\hbar)\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle. \quad (\text{Schrödinger equation})$$

$$\Rightarrow |\psi(t)\rangle = \underbrace{\mathcal{T}\left[e^{(-i/\hbar)\int_0^t \hat{H}(t')dt'}\right]}_{\hat{U}(t)}|\psi(0)\rangle$$

Q) Any unitary operator can be expressed as

$$\hat{U} = e^{i\hat{A}}, \quad \text{wher } \hat{A} \text{ is Hermitian.}$$

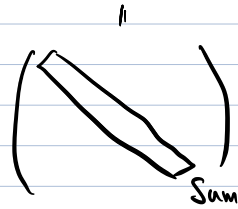
(c.f.)

$$e^{\hat{O}} = \mathbb{1} + \hat{O} + \frac{1}{2!}\hat{O}^2 + \cdots$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\hat{O}^n$$

• <u>Trace of operators</u>

$$\mathrm{Tr}[\hat{O}] = \sum_i \langle i|\hat{O}|i\rangle = \sum_i O_{ii}$$

$\parallel$



Sum of diagonal elements.

(3) Qubits.

Let's consider $\mathcal{H}^{d=2}$. $\binom{\alpha}{\beta}$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad // \quad \binom{\alpha}{\beta} \qquad |\alpha|^2+|\beta|^2 = 1$$

$$= \cos(\theta/2)|0\rangle + e^{i\varphi}\sin(\theta/2)|1\rangle \qquad \begin{cases} 0 \le \theta \le \pi \\ 0 \le \varphi \le 2\pi \end{cases}$$

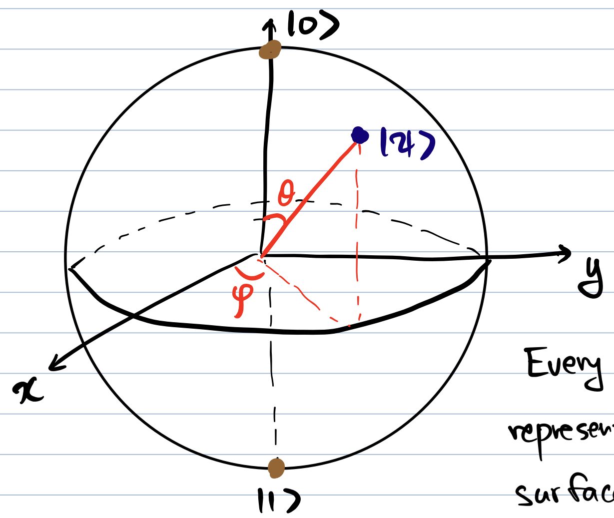
- Bloch sphere

Why we can take $0 \le \alpha \le 1$?

$\Rightarrow$ overall phase does not change $\langle \hat{A} \rangle_\psi$

i.e., $\langle\psi| e^{i\chi} \hat{A} e^{-i\chi} |\psi\rangle$

$= \langle\psi|\hat{A}|\psi\rangle$



Every $|\psi\rangle \in \mathcal{H}^2$ can be represented as a point on the surface of a sphere.

$\updownarrow$

Classical bit (0 or 1)

$\Rightarrow$ only two points (N/S poles)

$$\begin{cases} X|0\rangle = |1\rangle \\ X|1\rangle = |0\rangle \\ Z|0\rangle = |0\rangle \\ Z|1\rangle = (-1)|1\rangle \end{cases}$$

- Pauli matrix

$(X^2 = \mathbb{1},\ Y^2 = \mathbb{1},\ Z^2 = \mathbb{1})$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1) X, Y, Z are Hermitian. (& unitary) $\quad \hat{\mathbb{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2) Commutation relation: $[X, Y] = 2iZ$, $[Y, Z] = 2iX$, $[Z, X] = 2iY$

3) Any Hermitian operator can be expressed as

SUM

$$\hat{A} = d_0\hat{\mathbb{1}} + d_1 X + d_2 Y + d_3 Z \quad \text{with } \underline{d_i \in \mathbb{R}}.$$

4) $Tr[X] = Tr[Y] = Tr[Z] = 0$

" Q) $\hat{U} = e^{i(d_1 X + d_2 Y + d_3 Z)}$ (unitary operator)

- Two and more qubits

Hilbert space of two qubits : $\mathcal{H}^2 \otimes \mathcal{H}^2$ ← tensor product.

Basis : $\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$ (a.k.a. computational basis)

$(|i\rangle \otimes |j\rangle \equiv |ij\rangle$ or $|i\rangle|j\rangle)$

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix}$$

We can repeat this to construct a n-qubit system

$$|\psi\rangle = \left.\begin{pmatrix} \alpha_{0\cdots0} \\ \vdots \\ \alpha_{1\cdots1} \end{pmatrix}\right] 2^n \text{ (complex-valued) elements.}$$

(Size of Hilbert space grows exponentially)

Classical n-bit can be presented $\overbrace{010010\cdots1}^{n}$

(4) Pure and mixed states.

- How do we express classically random states?

(e.g. coin tossing)

Ⓗ Ⓣ .... Ⓗ Ⓗ ... Ⓣ $\left(\begin{matrix} p : \text{Head } (0) \\ 1-p : \text{Tail } (1) \end{matrix} \right)$

$|\psi\rangle = \sqrt{p}\,|0\rangle + \sqrt{1-p}\,|1\rangle$ is not a good choice

as the coin tossing is not superposition.

$\Rightarrow$ We use a density operator.

$$\hat{\rho} = p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$$

" General expression of quantum states (including classical mixtures) "

⓪ $\hat{\rho} = \hat{\rho}^\dagger$ (Hermitian)

① $\hat{\rho} \geq 0$ (positive semi definite)

② $\mathrm{Tr}[\hat{\rho}] = 1$. (trace is 1).

rank 1 operator

- How about $|\psi\rangle$? $\hat{\rho} = |\psi\rangle\langle\psi|$ (pure state)

- How do we calculate the expectation values?

$$\langle \hat{A} \rangle_{\hat{\rho}} = \mathrm{Tr}[\hat{A}\hat{\rho}]$$

(e.g.) For a pure state,

$$\begin{aligned} \mathrm{Tr}[\hat{A}\,|\psi\rangle\langle\psi|] &= \sum_i \langle i|\hat{A}|\psi\rangle\langle\psi|i\rangle \\ &= \sum_i \langle\psi|i\rangle\langle i|\hat{A}|\psi\rangle \\ &= \langle\psi|\left(\sum_i |i\rangle\langle i|\right)\hat{A}|\psi\rangle \\ &= \langle\psi|\hat{A}|\psi\rangle. \end{aligned}$$

- As $\hat{\rho}$ is a Hermitian operator, eigen-decomposition exists.

$$\hat{\rho} = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i| , \quad \text{where} \quad \langle\phi_i|\phi_j\rangle = \delta_{ij} \quad (\text{basis})$$

$$\& \ \lambda_i \in \mathbb{R}$$

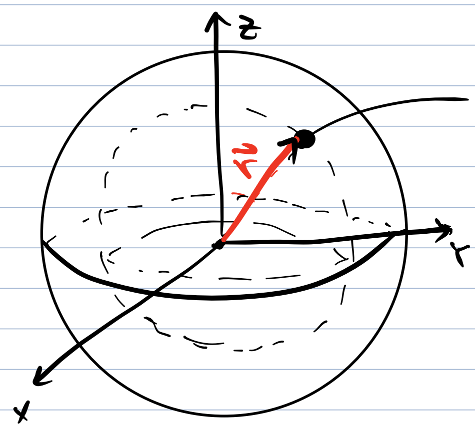
$$\begin{cases} \hat{\rho} \geq 0 \iff \lambda_i \geq 0 \quad \forall i \\ \mathrm{Tr}[\hat{\rho}] = 1 \iff \sum_i \lambda_i = 1 \end{cases}$$

(c.f.) $P_i = \mathrm{Tr}[\hat{P}_i \hat{\rho}]$

$0 \leq P_i \leq 1$ for all $\hat{P}_i$

$\Rightarrow \hat{\rho} \geq 0$

- Mixed state in a qubit system



$\hat{\rho}$: Point inside the Bloch sphere.

Using Pauli matrices,

$$\hat{\rho} = \frac{1}{2}\left(\mathbb{1} + r_x X + r_y Y + r_z Z\right)$$

with $\vec{r} = (r_x, r_y, r_z) \in \tilde{S}$

$\therefore$ For a pure state, $|\psi\rangle\langle\psi|$, $|\vec{r}| = 1$.

- How do we know whether a state is pure or mixed?
(rank)

$$\mathrm{Tr}[\rho^2] = \sum_i \lambda_i^2 \begin{cases} = 1 & (\text{pure state}) \\ < 1 & (\text{mixed state}) \end{cases}$$

$\uparrow$ purity

Some other functions? Entropy, ...

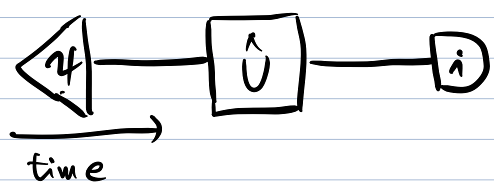
- Mixedness is also important to study "quantum entanglement"

(5) Quantum circuits.

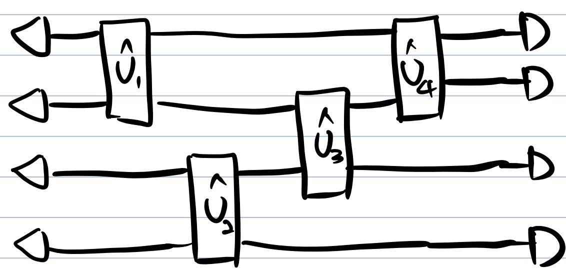
- state : $|\psi\rangle \longrightarrow \hat{\rho}$
- evolution : $\hat{U}|\psi\rangle \longrightarrow \hat{U}\hat{\rho}\hat{U}^\dagger$ (more general process will be studied soon)
- observables : $\langle\psi|\hat{A}|\psi\rangle \longrightarrow \mathrm{Tr}[\hat{A}|\psi\rangle\langle\psi|]$
- measurement : $|\langle i|\psi\rangle|^2 \longrightarrow \mathrm{Tr}[\underbrace{|i\rangle\langle i|}_{\hat{P}_i}\hat{\rho}]$

  $\hat{P}_i$ (projection operator)

To visualize this, we use quantum circuit representation.



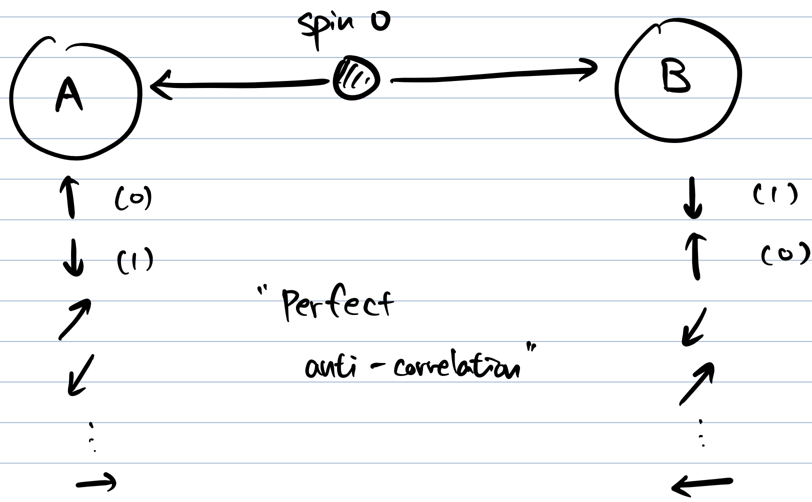
For a multi-qubit system,



II. Nonlocality and Entanglement.

(1) Two-qubit Bell state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \qquad \text{(Bell - state)}$$

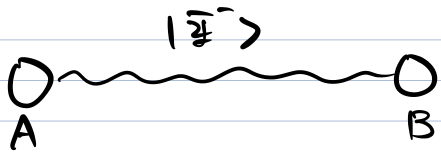


"What is special with such a correlation?"

(2) EPR Paradox & Bell's theorem

- EPR paradox (2 qubit version)

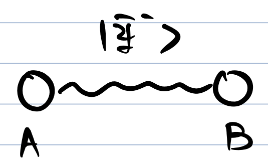
"Phys. Rev. 47, 777 (1935)" for original paper with $(\hat{x}, \hat{p})$



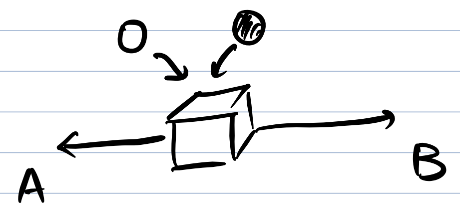
- By measuring B in ↑/↓ (z-axis), we know how A will behave (↓/↑) without measuring it.
- By measuring A in →/← (x-axis), we know A's state in the x-basis.
- However, X and Z of A cannot be simultaneously determined! (due to the uncertainty principle)

- Einstein's main question
  " Can we have a complete theory that can describe the outcomes from quantum mechanics? "
  → Local hidden variable theory.

- Bell's theorem.



vs.



- Local hidden variable " λ "

  For a given " λ ", measurement axis $(x, y, z, \cdots)$ $\begin{cases} P(a \mid M_a, \lambda) \\ P(b \mid M_b, \lambda) \end{cases}$ are fully determined.

$$\Rightarrow \quad P(a, b \mid M_a, M_b, \lambda) = P(a \mid M_a, \lambda)\, P(b \mid M_b, \lambda)$$

Then, statics that can be explained by LHV are

$$\langle O_A O_B \rangle = \underbrace{\int_\Lambda d\mu_\lambda}_{\text{(dist. of } \lambda)} \sum_{a,b} O_a O_b \, P(a, b \mid M_a, M_b, \lambda)$$

$$= \int_\Lambda d\mu_\lambda \left(\sum_a O_a P(a \mid M_a, \lambda) \right) \left(\sum_b O_b P(b \mid M_b, \lambda) \right)$$

$$= \int_\Lambda d\mu_\lambda \; E(A, \lambda) \cdot E(B, \lambda)$$

- Can a LHV model explain quantum mechanical prediction?
  ⇒ **No.** (Bell's theorem / inequality)

- CHSH inequality

For different measurements A, A' & B, B' with outcomes ± 1, (O_A, O_B)

$$B = \langle O_A O_B \rangle + \langle O_{A'} O_B \rangle + \langle O_{A'} O_{B'} \rangle - \langle O_A O_{B'} \rangle$$

$\Rightarrow \quad |B| \leq 2$ for "any LHV model."

(pf.)

$$|\langle O_A O_B \rangle - \langle O_A O_{B'} \rangle| = \left| \int_\Lambda d\mu_\lambda \, E(A,\lambda)\left(E(B,\lambda) - E(B',\lambda) \right) \right|$$

$$= \left| \int_\Lambda d\mu_\lambda \left[E(A,\lambda) E(B,\lambda) \underbrace{(1 \pm E(A',\lambda) E(B',\lambda))}_{\geq 0} - E(A,\lambda) E(B',\lambda) \underbrace{(1 \pm E(A',\lambda) E(B,\lambda))}_{\geq 0} \right] \right|$$

$$\leq \int_\Lambda d\mu_\lambda \left[\underbrace{|E(A,\lambda) E(B,\lambda)|}_{\leq 1} \cdot (1 \pm E(A',\lambda) E(B',\lambda)) + |E(A,\lambda) E(B',\lambda)| \cdot (1 \pm E(A',\lambda) E(B,\lambda)) \right]$$

$$\leq 2 \pm \langle O_{A'} O_{B'} \rangle \pm \langle O_{A'} O_B \rangle$$

$$\leq 2 - |\langle O_{A'} O_{B'} \rangle + \langle O_{A'} O_B \rangle|$$

$$\Rightarrow |B| \leq |\langle O_A O_B \rangle - \langle O_A O_{B'} \rangle| + |\langle O_{A'} O_{B'} \rangle + \langle O_{A'} O_B \rangle|$$

$$\leq 2$$

- Quantum mechanical violation of the CHSH inequality

For $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

$$\langle O_A O_B \rangle = \langle \Psi^- | \hat{O}_A \hat{O}_B | \Psi^- \rangle.$$

Q) Show that $\hat{O}_{A,A',B,B'}$ has eigenvalues ± 1

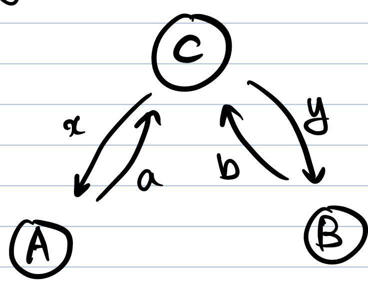
By taking
$$\begin{cases} \hat{O}_A = X & \hat{O}_{A'} = Z \\ \hat{O}_B = -\frac{X+Z}{\sqrt{2}}, & \hat{O}_{B'} = \frac{X-Z}{2} \end{cases}$$

$\Rightarrow$ $\underline{\underline{B = 2\sqrt{2} > 2}}$!

Bell inequality violated!

(3) Nonlocal quantum game

- CHSH game



$x, y \in \{0,1\}$
$a, b \in \{0,1\}$

1. Alice and Bob win the game when $\overset{a+b \pmod 2}{\overset{\shortparallel}{a \oplus b}} = x \cdot y$.
2. Alice and Bob can decide their strategy before entering the game.
3. Alice and Bob cannot communicate after entering the game

$\Rightarrow$ What is the highest winning probability & strategy?

Probability to distribut (x,y). here $P_x = P_y = 1/2 \Rightarrow P(x,y) = 1/4$.

$$P_{win} = \sum_{a,b,x,y} p(x,y)\, \underbrace{P(a,b|x,y)}_{\text{Alice \& Bob's strategy. (Conditional probability)}}\, \underbrace{\delta_{a\oplus b,\, x\cdot y}}_{\text{Pay-off function}}$$

$$= \frac{1}{4} \sum_{a \oplus b = x\cdot y} P(a,b|x,y)$$

$$= \frac{1}{4}\Big[P(0,0|0,0) + P(1,1|0,0) + P(0,0|0,1) + P(1,1|0,1) + P(0,0|1,0) + P(1,1|1,0) + \underbrace{P(0,1|1,1) + P(1,0|1,1)}_{= 1 - P(0,0|1,1) - P(1,1|1,1)} \Big]$$

$$= \frac{1}{4}\Big[P(0,0|0,0) + P(1,1|0,0) + P(0,0|0,1) + P(1,1|0,1) + P(0,0|1,0) + P(1,1|1,0) - P(0,0|1,1) - P(1,1|1,1) + 1 \Big]$$

$$= \frac{1}{4}\Big[P(a=b|0,0) + P(a=b|0,1) + P(a=b|1,0) - P(a=b|1,1) + 1 \Big]$$

Let's define $E(x,y) = (+1)P(a=b|x,y) + (-1)\underbrace{P(a\neq b|x,y)}_{=1-P(a=b|x,y)}$

$$= 2P(a=b|x,y) - 1$$

$$\therefore P(a=b|x,y) = \frac{E(x,y)+1}{2}$$

Then,

$$P_{win} = \frac{1}{4}\left[\frac{E(0,0)}{2} + \frac{E(0,1)}{2} + \frac{E(1,0)}{2} - \frac{E(1,1)}{2} + 2\right]$$

$$= \frac{B+4}{8}$$

Bell value !

$$\leq 3/4 \quad \text{(Classically possible strategy)}$$

But if Alice and Bob shares $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$,

$P_{win} = \frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right) \approx 85\%$ (10% higer than classical strategy)

- Many other types of non local games.

"Quantum Prisoner's Dilemma"

Phys. Rev. Lett. 83, 3077 (1999)

(4) Quantum Entanglement.

- What makes these non-local behaviors?

$\Rightarrow$ Entanglement.

- $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \rightarrow$ (Maximally) entangled state.

  a.k.a. Bell state.

- Are there any other states with $B = 2\sqrt{2}$ (maximum)?

$$\left.\begin{cases} |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{cases}\right) : \underline{\text{Bell-States}}$$

- **Schmidt decomposition**

  Any bipartite pure state $|\Psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be expressed as

  orthogonal states in each subspace $\mathcal{H}_A / \mathcal{H}_B$

$$|\Psi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i}\, |\phi_i\rangle_A |\psi_i\rangle_B, \quad \text{where } d = \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$$

  with $\lambda_i \geq 0$: Schmidt coefficients.

  · Schmidt rank = # of non-zero λ_i's.

$\Rightarrow |\Psi\rangle_{AB}$ is $\begin{cases} \text{entangled (if Schmidt rank} > 1) \\ \text{separable (if Schmidt rank} = 1) \end{cases}$

(i.e., $|\Psi\rangle = |\phi\rangle_A \otimes |\psi\rangle_B$

$\Rightarrow$ How to find it? $\Rightarrow$ Singular value decomposition.

$$|\Psi\rangle_{AB} = \sum_{i,j} C_{ij} |i\rangle|j\rangle \Rightarrow |\Psi\rangle_{AB} = \sum_{i} \sqrt{\lambda_i}\, |\phi_i\rangle_A |\psi_i\rangle_B$$

$$C = U \Sigma V = U \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_d} \end{pmatrix} V$$

$$\Rightarrow C_{ij} = \sum_{k,l} U_{ik} \Sigma_{kl} V_{kj} = \sum_{k} U_{ik} V_{kj} \sqrt{\lambda_k}$$

$$\Rightarrow |\Psi\rangle_{AB} = \sum_{i,j,k} U_{ik} V_{kj} \sqrt{\lambda_k}\, |i\rangle|j\rangle$$

$$= \sum_{k} \sqrt{\lambda_k} \underbrace{\left(\sum_{i} U_{ik} |i\rangle\right)}_{= |\phi_k\rangle} \underbrace{\left(\sum_{j} V_{kj} |j\rangle\right)}_{= |\psi_k\rangle}$$

$$= \sum_{k} \sqrt{\lambda_k}\, |\phi_k\rangle|\psi_k\rangle. \qquad \blacksquare$$

Q) $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |00\rangle\right)$ entangled?

$$|\Psi\rangle = \sqrt{\frac{3}{5}}\, |00\rangle + \sqrt{\frac{1}{5}}\, |10\rangle + \sqrt{\frac{1}{5}}\, |11\rangle$$

$\Rightarrow$ What is the Schmidt decomposition?

- What can we do with entangle ment?

(c.f.) Quantum cryptography (AK91 protocol)

1) Super dense coding

" All Bell states are interconvertible via local unitaries "

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

X_A on $|\Phi^+\rangle$: $(X_A \otimes \mathbb{1}_B)|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$

$\begin{pmatrix} X|0\rangle = |1\rangle \\ X|1\rangle = |0\rangle \end{pmatrix}$ "bit flip" $\qquad = |\Psi^+\rangle$

Z_A on $|\Phi^+\rangle$: $(Z_A \otimes \mathbb{1}_B)|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

$\begin{pmatrix} Z|0\rangle = |0\rangle \\ Z|1\rangle = (-1)|1\rangle \end{pmatrix}$ "phase flip" $\qquad = |\Phi^-\rangle$

$\underbrace{Z_A X_A}_{= iY_A}$ on $|\Phi^+\rangle$: $(Z_A X_A) \otimes |\Phi^+\rangle = \frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle)$

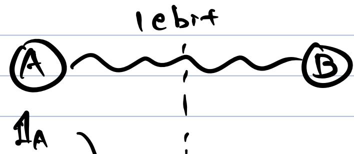
$= |\Psi^-\rangle$

- We can send 2 bit (00, 01, 10, 11)

using 1 Bell state (= 1 ebit) with 1 qubit communication

1 ebit

① Apply

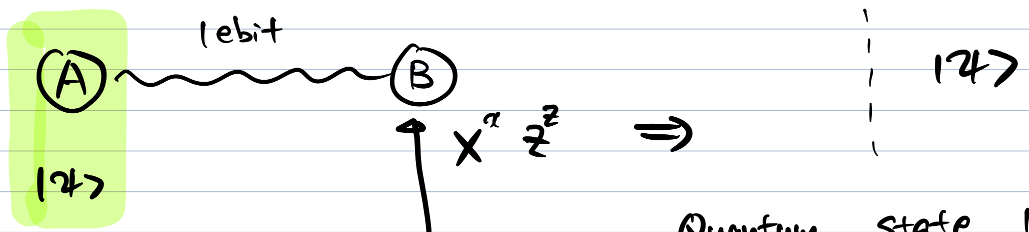


$\begin{pmatrix} 00 : & \mathbb{1}_A \\ 01 : & X_A \\ 10 : & Z_A \\ 11 : & Y_A \end{pmatrix}$

② send A's qubit to B

③ Bob checks his state

2) Quantum teleportation.



Joint - measurement on $\{|\Phi^{\pm}\rangle, |\Psi^{\pm}\rangle\}$

(00, 01, 10, 11)

x z

Quantum state $|\psi\rangle$ appears at Bob!

2bit + 1 ebit $\Rightarrow$ 1 qubit communication

$\therefore$ Sharing as many copies of ebits as possible

$\Rightarrow$ Higher Q. communication Capacity.

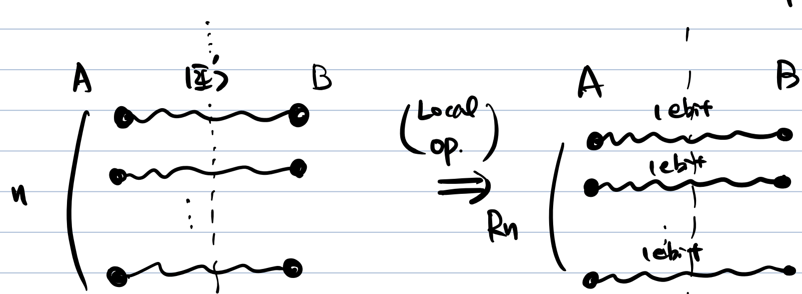
3) Quantum state distillation

- What happens if we have imperfect entanglement?

ex) $|\Phi'\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{\sqrt{2}}{\sqrt{3}}|11\rangle \Rightarrow E = 0.918$

$\Rightarrow$ We cannot directly use for teleportation,

but we can "distill" it to "perfect" entanglement.



n-copies of Φ' Rn-copies of Φ $\Rightarrow$ "Rn ebits"

We will learn more about this in the next lecture.

(5) Entanglement of mixed two-qubit states.

- $\hat{\rho}_{AB}$: mixed quantum state

is $\begin{cases} \text{Separable if } \hat{\rho}_{AB} = \sum_k P_k \, \hat{\rho}_A^{(k)} \otimes \hat{\rho}_B^{(k)} \\ \text{Entangled if not separable.} \end{cases}$

- In general, for $d \gg 1$, it is very difficult to determine whether $\hat{\rho}_{AB}$ is entangled or not.

(Peres & Horodecki)

- PPT (positive Partial trace) criteria.

" $\hat{\rho}_{AB}$ is entangled if $\hat{\rho}_{AB}^{T_B} \geqslant 0$ "

$$\hat{\rho}_{AB}^{T_B} = \Big(\sum_{\substack{i,j \\ k,l}} \rho_{ijkl} \, |i\rangle_A\langle j| \otimes \underbrace{|k\rangle_B\langle l|}_{\text{exchange}} \Big)^{T_B}$$

$$= \Big(\sum_{\substack{i,j \\ k,l}} \rho_{ijkl} \, |i\rangle_A\langle j| \otimes |l\rangle_B\langle k| \Big)$$

If $\hat{\rho}_{AB}$ is separable, $\hat{\rho}_{AB}^{T_B} \geqslant 0$.

(p.f.)

$$\hat{\rho}_{AB}^{T_B} = \Big(\sum_k P_k \, \hat{\rho}_A^{(k)} \otimes \hat{\rho}_B^{(k)} \Big)^{T_B}$$

$$= \sum_k P_k \, \underbrace{\hat{\rho}_A^{(k)}}_{\geqslant} \otimes \underbrace{\left(\hat{\rho}_B^{(k)}\right)^{T_B}}_{\geqslant 0}$$

$\left(\begin{array}{l} \text{if } \rho \geqslant 0 \\ \Rightarrow \rho^T \geqslant 0 \end{array} \right)$

$\geqslant 0.$

$\therefore$ If $\hat{\rho}_{AB}^{T_B} \ngeq 0$ (has negative eigenvalues) $\Rightarrow$ $\hat{\rho}_{AB}$ is entangled!

(e.g.) $\hat{\rho}_{AB} = |\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow \hat{\rho}_{AB}^{T_B} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

"negative"

$\mathrm{eig}(\hat{\rho}_{AB}^{T_B}) = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$

$\therefore \hat{\rho}_{AB} = |\Phi^+\rangle\langle\Phi^+|$ is entangled.

Q) Entanglement of a noisy quantum state.

$\hat{\rho}_w = \lambda|\Phi^+\rangle\langle\Phi^+| + (1-\lambda)\mathbb{1}_{AB}, \quad 0 \leq \lambda \leq 1.$

What is the minimum value of λ to have entanglement?

" PPT criteria is necessary & sufficient condition when $(\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)) = (2,2)$ and $(2,3)$"

" In a higher dimensional system, there exist entangled quantum states $(\hat{\rho}_{AB})$ with positive partial transpose $(\hat{\rho}_{AB}^{T_B} \geq 0)$"

- NPT bound entanglement problem

"Is any non-distillable entangled state PPT?"

(One of the most popular open problems in Q.I.)