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Stabilizer Formalism II

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Nielsen and Chuang, Chapter 10.5.5 ~ 10.5.8

Stabilizer Code Construction

Stabilizer Code

Pauli gate acting on the qubit *i*

- **Pauli group** over *n* qubits is defined as

$$\mathcal{P}_n = \{\alpha \times p_1 \otimes p_2 \otimes \cdots \otimes p_n |\ \alpha \in \{\pm 1, \pm i\},\ \boxed{p_i} \in \mathcal{P}_1 = \{I, X, Z, Y\}\},$$

  where each element of $\mathcal{P}_1$ is defined as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, and\ Y = iXZ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- **Stabilizer code space $\boldsymbol{C}$**: vector space spanned by the normalized *n*-qubit complex vectors.
- Each complex vector is fixed (stabilized) by the **stabilizer group $\boldsymbol{\mathcal{S}}$**.
- Stabilizer group is an *abelian* subgroup of $\boldsymbol{\mathcal{P}_n}$.

“$a \cdot b = b \cdot a$”

- Stabilizer group can be fully generated by a **minimal set of generators (*stabilizer generators $\mathcal{G}$*)**.

independent

Stabilizer Generators and Code Space

- **[[*n*,*k*]] stabilizer code** is defined by *n-k* independent stabilizer generators. $g_i \in \mathcal{P}_n$
- Stabilizer generators have to satisfy the following two conditions,
  - **(commutativity)** $g_i \cdot g_j = g_j \cdot g_i, \forall g_i \text{ and } \forall g_j \in \mathcal{G}$
  - **(code space)** $-I \notin \mathcal{G}$

$\rightarrow -1 \times I^{\otimes n}$

- **Stabilizer code space** $\mathcal{C}$ with respect to $\mathcal{G}$ is defined as

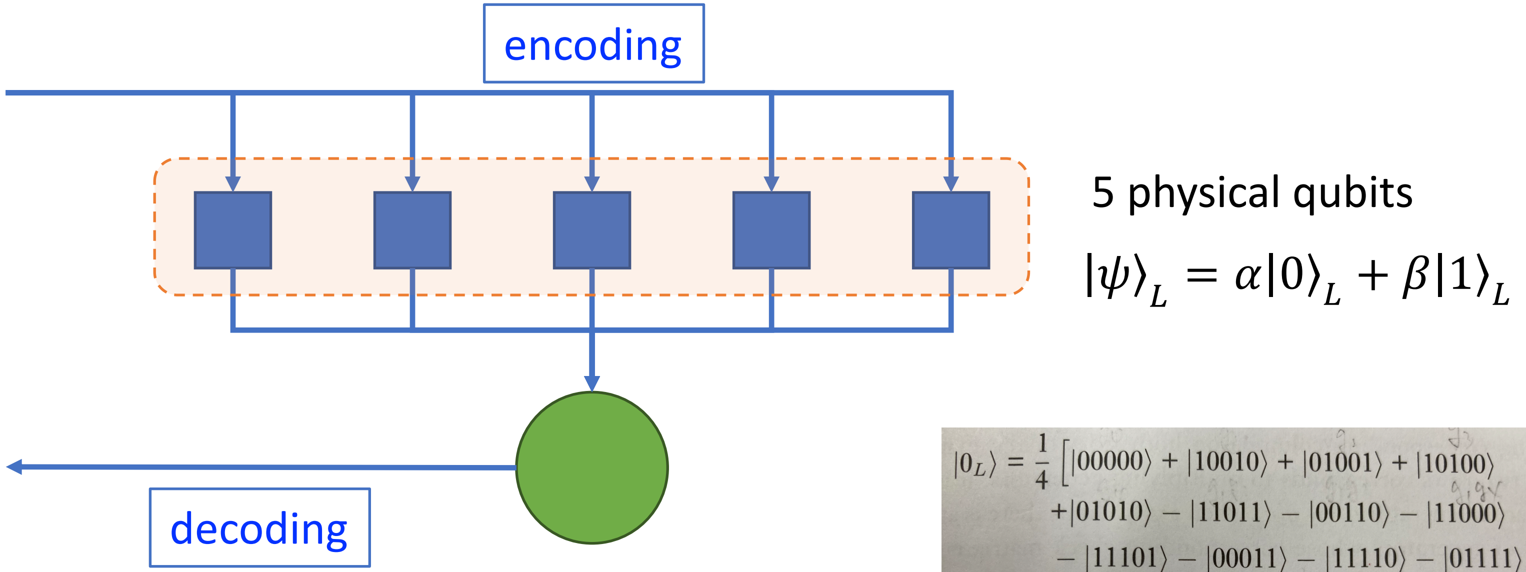
$$\mathcal{C} = span\,(\{\,|\psi\rangle \mid g_i|\psi\rangle = |\psi\rangle, \forall g_i \in \mathcal{G}\})$$

- Every codeword is a common eigenvector of all stabilizer generators $\mathcal{G}$ with eigenvalue +1

[[n,k]] Stabilizer Code (1/2)

- An [[n,k]] stabilizer code encodes k-qubit quantum information into n physical qubits.
  - For example,
    [[5,1,3]] stabilizer code:

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Code space : 2[1]-dimensional subspace of 2[5]-dimensional vector space

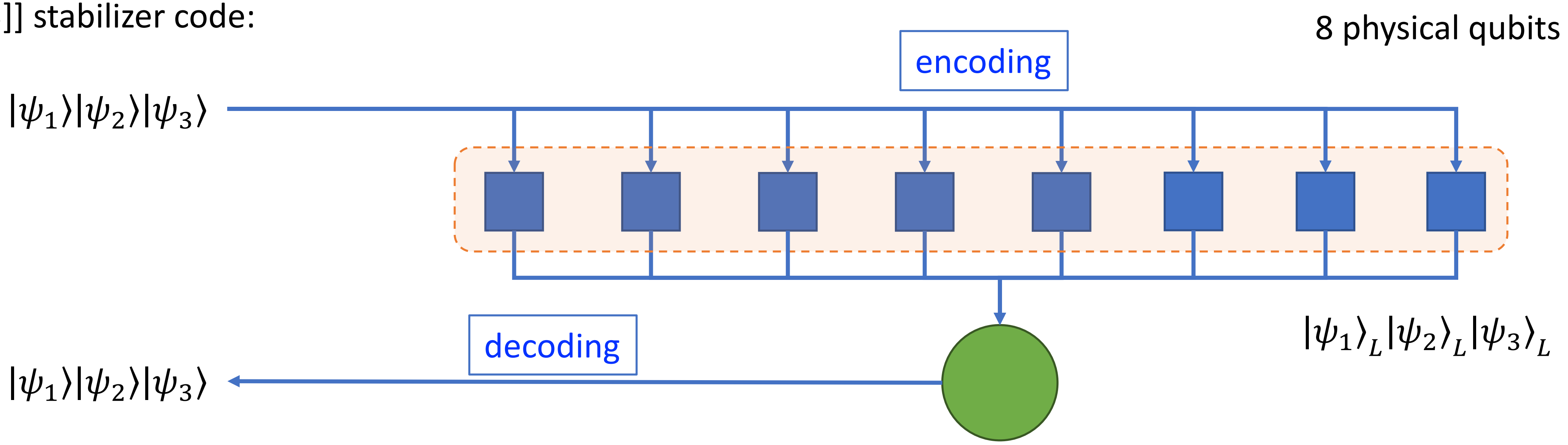
→ Elements: linear superposed vectors based on

$$|0_L\rangle = \frac{1}{4}\Big[|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle - |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle\Big]$$

$$|1_L\rangle = \frac{1}{4}\Big[|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle - |00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle\Big]$$

[[n,k]] Stabilizer Code (2/2)

- An [[n,k]] stabilizer code encodes k-qubit quantum information into n physical qubits.
  - For example,
    [[8,3,3]] stabilizer code:



Code space : 2^3-dimensional subspace of 2^8-dimensional vector space

→ Elements: 2^1-dim subspace for $|\psi_i\rangle_L$ where $i = 1,2,3$

Basis

- Basis of a physical qubit: $|0\rangle$ and $|1\rangle$. $Z|0\rangle = |0\rangle$, $X|0\rangle = |1\rangle$

- **Logical Basis** for an [[*n*, *k*]] stabilizer code:
  - *k*=1:
    - $|0\rangle_L$ and $|1\rangle_L$, $|\psi\rangle_L = \alpha|0\rangle_L + \beta|1\rangle_L$
    - $g_i|\psi\rangle_L = |\psi\rangle_L$, $\forall g_i \in \mathcal{G}$ and $Z_L|0\rangle_L = |0\rangle_L$, $X_L|0\rangle_L = |1\rangle_L$

$$\langle g_1, \cdots, g_{n-k}, Z_L^1, \cdots, Z_L^k \rangle$$

  - *k*>1:
    - $|0\rangle_L^i$ and $|1\rangle_L^i$ for $i = 1,2,..,k$. $|\psi\rangle_L^i = \alpha|0\rangle_L^i + \beta|1\rangle_L^i$
    - $g_i|\psi\rangle_L^i = |\psi\rangle_L^i$, $\forall g_i \in \mathcal{G}$
    - $Z_L^i|0\rangle_L^i = |0\rangle_L^i$ and $X_L^i|0\rangle_L^i = |1\rangle_L^i$

Stabilizer Code vs Classical Linear Code

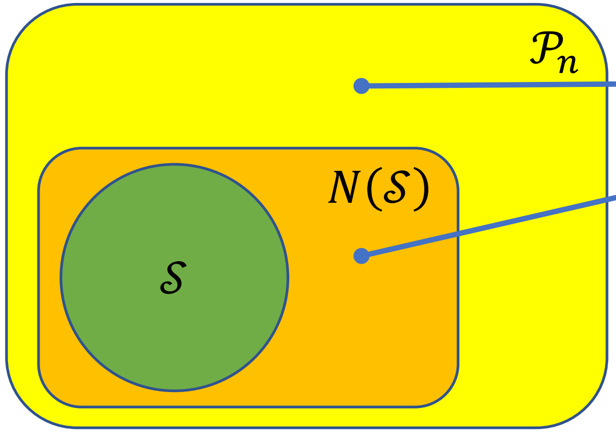
- Stabilizer code is quantum analogue of classical linear code.
- (Linear code) Codewords are generated by generator matrix and stabilized by parity check matrix
- (Stabilizer code) Codewords are generated and stabilized by stabilizer (→ quantum parity check matrix)

Linear Code		Stabilizer Code
parity check matrix	Error Checker	stabilizer (stabilizer generator)
$H \in \mathbb{F}_2^{r \times n}$		$\mathcal{S} = \langle g_1, .., g_r \rangle, g_i \in \mathcal{P}_n(\mathbb{F}_2^{1 \times 2n})$
$H \cdot G = 0$	Condition	$g_i \cdot g_j = g_j \cdot g_i$
$y = Gx$, x: information, y: codeword	Encoding	$\|0\rangle_L = \prod(1 + g_i)\|0 \cdots 0\rangle$
$\tilde{y} = y + e, e \in \mathbb{F}_2^n$	Erroneous Codeword	$\widetilde{\|\psi\rangle_L} = U(I + e)\ \|\psi\rangle_L, e \in \mathcal{P}_n$
$H\tilde{y} = s$	Syndrome	$g_i \widetilde{\|\psi\rangle_L}\|0\rangle_a = \widetilde{\|\psi\rangle_L}\|s\rangle_a$

Normalizer

- **Stabilizer** $\mathcal{S}$ is an abelian subgroup of $\mathcal{P}_n$.
- For stabilizer $\mathcal{S}$,
  - **Normalizer** of $\mathcal{S}$, $N(\mathcal{S})$, is a set of an operator $\mathcal{E}$ such that $\underline{\mathcal{E} g \mathcal{E}^\dagger \in \mathcal{S}}$ for all $g \in \mathcal{G}$.
  - $\mathcal{S} \subset N(\mathcal{S}) \subset \mathcal{P}_n$

“$\mathcal{E} g = g \mathcal{E}$”



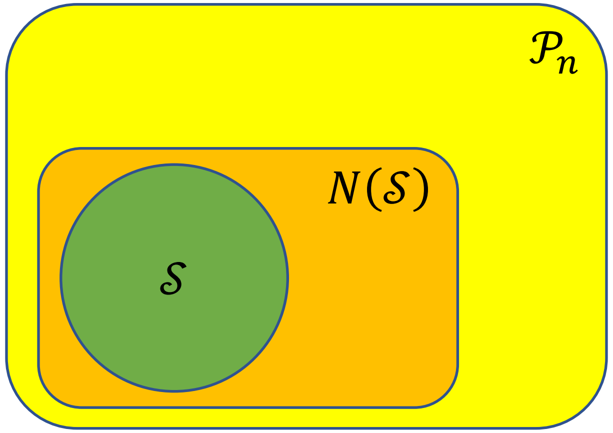
anti-commutable with $\mathcal{S}$, $\{g, E\} = 0$

commutable with $\mathcal{S}$, $[g, E] = 0$

→ If an element in $N(\mathcal{S})$ occurs to a stabilizer codeword as an error, it can not be detected !!

Error Correction Condition

- Let $\mathcal{S}$ be the stabilizer for a stabilizer code $\mathcal{C}$. Suppose that $\{\mathcal{E}_j\}$ is a set of operators in $\mathcal{P}_n$ such that $\mathcal{E}_j^\dagger \mathcal{E}_{\boldsymbol{k}} \notin N(\mathcal{S}) - \mathcal{S}$ for all *j* and *k*. Then, $\{\mathcal{E}_j\}$ is a correctable set of errors for the code $\mathcal{C}$.



- **Detectable** error : $\mathcal{E} \in \mathcal{S}$ or $\mathcal{E} \in \mathcal{P}_n - N(\mathcal{S})$
- **Undetectable** error : $\mathcal{E} \in N(\mathcal{S}) - \mathcal{S}$

  manipulate the stabilizer codewords
  (→ ***logical operator of Stabilizer code***)

- Code distance: the minimum weight of an element in $\underline{N(\mathcal{S}) - \mathcal{S}}$

  Undetectable error
  Logical operator

For [[7,1,3]] Steane code (d=3), $Z_L = Z_2 Z_4 Z_6$ (Nielsen and Chuang)

Binary vector representation of Pauli operator

- 1-qubit Pauli operator $\mathcal{P}_1$ can be represented by a 2-bit vector (without phase information)
  $I = (0|0), X = (1|0), Z = (0|1)$ and $Y = (1|1)$
  $I = X^0Z^0, X = X^1Z^0, Z = X^0Z^1$ and $Y = X^1Z^1$

- *n*-qubit Pauli operator $\mathcal{P}_n$ is represented by a 2*n*-bit vector.
  For example, $XZZXI = (10010|01100)$

1st qubit 4-th qubit

Matrix Representation of Stabilizer Code

- *n-k* Stabilizer written in Pauli operator can be represented by binary matrix as follows

$$\mathcal{G} = \left[\begin{array}{c|c} G_X & G_Z \end{array}\right]$$

  where G_x and G_z are binary matrices of size *n-k* x *n.*

- Note that the matrices G_x and G_z satisfy the **symplectic inner product condition**,

$$G_X \cdot G_Z^T + G_Z \cdot G_X^T = 0 \pmod 2$$

- For example,

$$\mathcal{G}^{[[5,1,3]]} = \begin{pmatrix} X & Z & Z & X & I \\ I & X & Z & Z & X \\ X & I & X & Z & Z \\ Z & X & I & X & Z \end{pmatrix} \rightarrow \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Examples

- **Developing a stabilizer code** equals to **find a pair of two binary matrices** satisfying the SIP condition.
- There exist quantum versions of BCH, Reed-Solomon, Reed-Muller, Convolutional, LDPC, Polar, .., by properly modifying the parity check matrices of the classical codes.

$$G_x = \begin{pmatrix} 1\,0\,1 & 0\,1\,0\,1 & 0\,1\,0\,1 & 0\,1\,0\,1 \\ 0\,1\,1 & 0\,0\,1\,1 & 0\,0\,1\,1 & 0\,0\,1\,1 \\ 0\,0\,0 & 1\,1\,1\,1 & 0\,0\,0\,0 & 1\,1\,1\,1 \\ 0\,0\,0 & 0\,0\,0\,0 & 1\,1\,1\,1 & 1\,1\,1\,1 \end{pmatrix} \quad G_z = \begin{pmatrix} 1\,0\,0 & 0\,1\,0\,0 & 0\,1\,0\,0 & 0\,1\,0\,0 \\ 0\,1\,0 & 0\,0\,1\,0 & 0\,0\,1\,0 & 0\,0\,1\,0 \\ 0\,0\,1 & 0\,0\,0\,1 & 0\,0\,0\,1 & 0\,0\,0\,1 \\ 0\,0\,0 & 1\,0\,1\,0 & 0\,0\,0\,0 & 1\,0\,1\,0 \\ 0\,0\,0 & 0\,1\,0\,1 & 0\,0\,0\,0 & 0\,1\,0\,1 \\ 0\,0\,0 & 0\,0\,1\,1 & 0\,0\,0\,0 & 0\,0\,1\,1 \\ 0\,0\,0 & 0\,0\,0\,0 & 1\,0\,1\,0 & 1\,0\,1\,0 \\ 0\,0\,0 & 0\,0\,0\,0 & 0\,1\,0\,1 & 0\,1\,0\,1 \\ 0\,0\,0 & 0\,0\,0\,0 & 0\,0\,1\,1 & 0\,0\,1\,1 \\ 0\,0\,0 & 0\,0\,0\,0 & 0\,0\,0\,0 & 1\,1\,1\,1 \end{pmatrix} \Rightarrow G_X G_Z^T + G_Z G_X^T = 0$$

[[15,1,3]] RM Code

Revisit of QECC in terms of Stabilizer Formalism

2-qubit Bell State

- State Representation

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$$

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle \pm |11\rangle\Big),\ |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}\Big(|01\rangle \pm |10\rangle\Big)$$

$$X_1X_2|\Phi^+\rangle = X_1X_2\frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$$

$$Z_1Z_2|\Phi^+\rangle = Z_1Z_2\frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$$

- Stabilizer Generators

$$\mathcal{G} = \langle X_1X_2, Z_1Z_2\rangle$$

$$g|\Phi^+\rangle = |\Phi^+\rangle,\ \forall g \in \mathcal{S} \text{ generated by } \mathcal{G}$$

3-qubit Cat State

- State Representation

$$|cat\rangle = \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big)$$

$$X_1X_2X_3|cat\rangle = X_1X_2X_3\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big)$$

$$Z_1Z_2|cat\rangle = Z_1Z_2\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big)$$

$$Z_2Z_3|cat\rangle = Z_2Z_3\frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big)$$

- Stabilizer Generators

$$\mathcal{G} = \langle X_1X_2X_3, Z_1Z_2, Z_2Z_3\rangle$$
$$g|cat\rangle = |cat\rangle,\ \forall g \in \mathcal{S} \text{ generated by } \mathcal{G}$$

n-qubit Cat State

- State Representation

$$|n-cat\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\Big)$$

$$X^{\otimes n}|n-cat\rangle = X^{\otimes n}\frac{1}{\sqrt{2}}\Big(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\Big) = \frac{1}{\sqrt{2}}\Big(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\Big)$$

$$Z_iZ_{i+1}|n-cat\rangle = Z_iZ_{i+1}\frac{1}{\sqrt{2}}\Big(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\Big) = \frac{1}{\sqrt{2}}\Big(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\Big), \text{for } i = 1\cdots n-1$$

- Stabilizer Generators

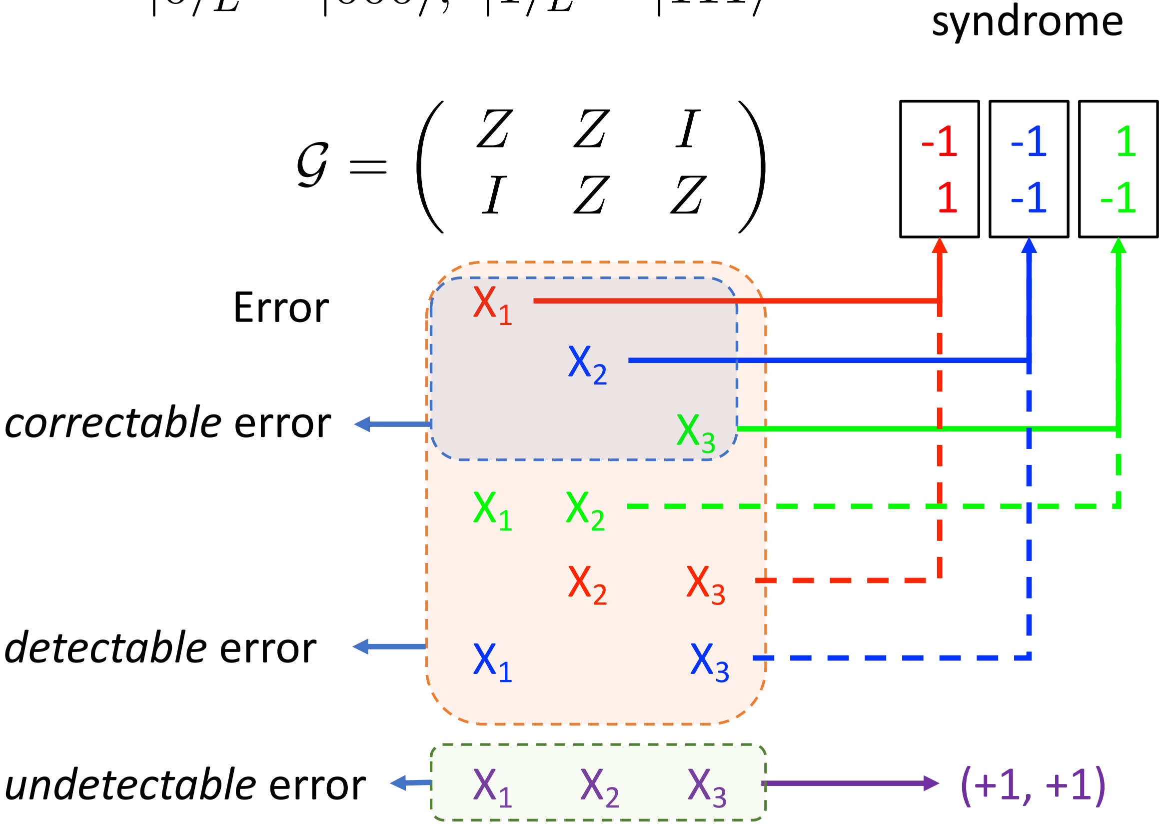
$$\mathcal{G} = \langle X_1 \cdots X_n, Z_1Z_2, \cdots, Z_{n-1}Z_n\rangle$$
$$g|n-cat\rangle = |n-cat\rangle, \forall g \in \mathcal{S} \text{ generated by } \mathcal{G}$$

3-qubit Bit-Flip/Phase-Flip Code

- Bit-Flip (X) Code

$$|0\rangle_L = |000\rangle, \ \ |1\rangle_L = |111\rangle$$

$$\mathcal{G} = \begin{pmatrix} Z & Z & I \\ I & Z & Z \end{pmatrix}$$

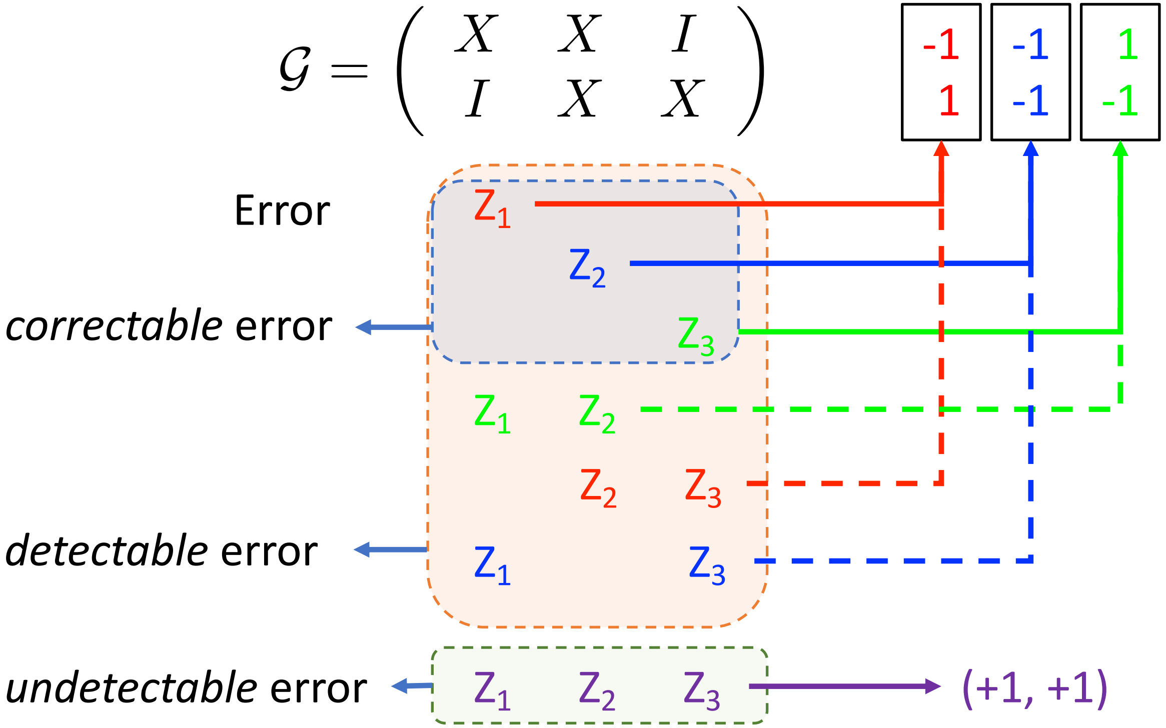


$$|\psi\rangle_L = \alpha|0\rangle_L + \beta|1\rangle_L \rightarrow \alpha|1\rangle_L + \beta|0\rangle_L$$

- Phase-Flip (Z) Code

$$|0\rangle_L = |+++\rangle, \ \ |1\rangle_L = |---\rangle$$

$$\mathcal{G} = \begin{pmatrix} X & X & I \\ I & X & X \end{pmatrix}$$



$$|\psi\rangle_L = \alpha|0\rangle_L + \beta|1\rangle_L \rightarrow \alpha|1\rangle_L + \beta|0\rangle_L$$

[[5,1,3]] Code

- The minimum code for correcting arbitrary 1-qubit error (quantum singleton bound)

$$n - k \geq 2(d-1)$$

$$\mathcal{G} = \begin{pmatrix} X & Z & Z & X & I \\ I & X & Z & Z & X \\ X & I & X & Z & Z \\ Z & X & I & X & Z \end{pmatrix}, \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

[[9,1,3]] Shor Code

- **Concatenation** of **Bit-Flip Code** and **Phase-Flip Code**

$$\mathcal{G} = \begin{pmatrix} Z & Z & I & I & I & I & I & I & I \\ I & Z & Z & I & I & I & I & I & I \\ I & I & I & Z & Z & I & I & I & I \\ I & I & I & I & Z & Z & I & I & I \\ I & I & I & I & I & I & Z & Z & I \\ I & I & I & I & I & I & I & Z & Z \\ X & X & X & X & X & X & I & I & I \\ I & I & I & X & X & X & X & X & X \end{pmatrix}$$

$$|0\rangle_L = \frac{1}{2\sqrt{2}}\big(|000\rangle + |111\rangle\big)\big(|000\rangle + |111\rangle\big)\big(|000\rangle + |111\rangle\big)$$

$$|1\rangle_L = \frac{1}{2\sqrt{2}}\big(|000\rangle - |111\rangle\big)\big(|000\rangle - |111\rangle\big)\big(|000\rangle - |111\rangle\big)$$

- The location of 1-qubit phase flip (Z) error cannot be identified, but the codeword can be successfully recovered.
  For example, Z_1 and Z_2 (or Z_3) leads to the same error syndromes.

[[7,1,3]] Steane Code

- [[7,1,3]] Steane code based on a pair of [7,4,3] Hamming codes

$$\mathcal{G}^{[[7,1,3]]} = \begin{pmatrix} I & I & I & X & X & X & X \\ I & X & X & I & I & X & X \\ X & I & X & I & X & I & X \\ I & I & I & Z & Z & Z & Z \\ I & Z & Z & I & I & Z & Z \\ Z & I & Z & I & Z & I & Z \end{pmatrix}$$

detecting phase-flip error

detecting bit-flip error

$\begin{bmatrix} H & O \\ O & H \end{bmatrix}$ where $H = \boxed{\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}}$

parity check matrix of [7,4,3] Hamming code

CSS Code

- Suppose C_1 and C_2 are [n, k_1] and [n,k_2] classical linear codes such that $C_2 \subset C_1$ and C_1 and $C_2^{\perp}$ both correct *t* errors. Then, CSS code can be defined with the following stabilizer (matrix form).

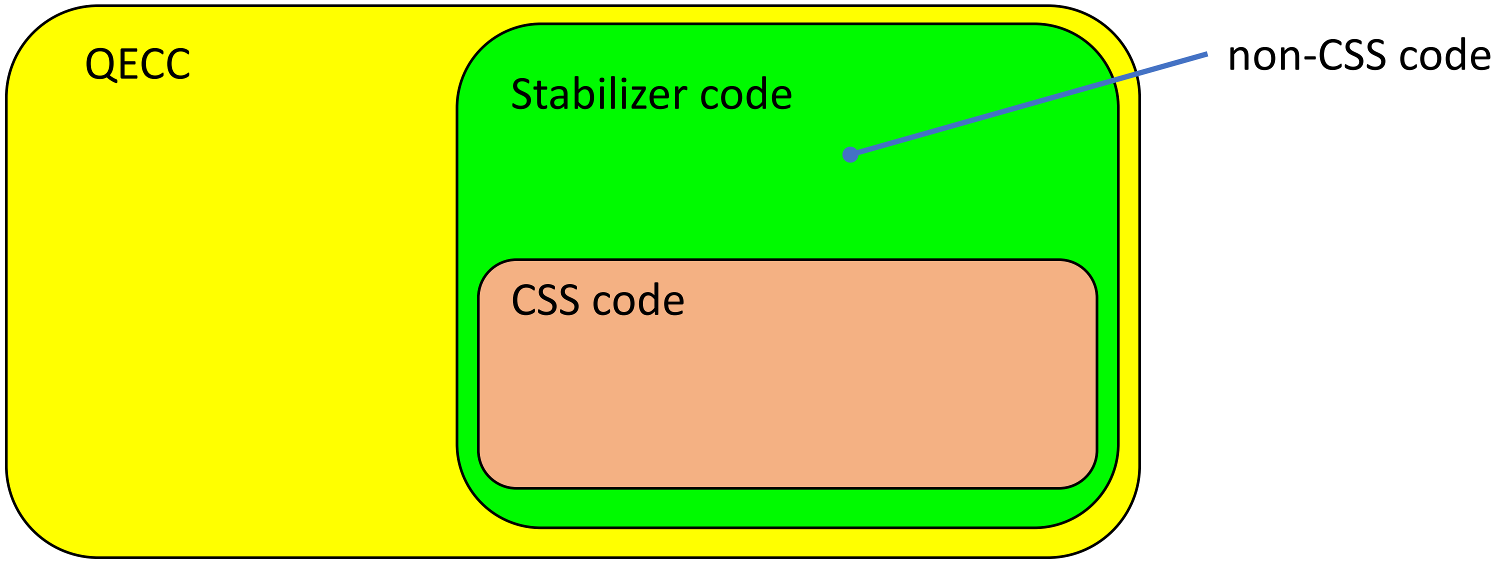
$$\left[\begin{array}{c|c} H(C_2^{\perp}) & 0 \\ 0 & H(C_1) \end{array}\right] \text{ with } H(C_2^{\perp}) \times H(C_1)^T = 0$$

- If $C_1 = C_2^{\perp}$, then the code is called *self-dual.*
- For example, [[7,1,3]] Steane code is defined the stabilizer

$$\left[\begin{array}{c|c} H & 0 \\ 0 & H \end{array}\right], \qquad \text{where } H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

CSS code and non-CSS code

- Note that each stabilizer operator of a **CSS code** is composed of solely X or Z (see [[7,1,3]] code or [[9,1,3]] code).
  On the contrary, **non-CSS code** (stabilizer code outside class of CSS codes) is defined by stabilizers composed of mixed X and Z (see [[5,1,3]] code).



<Hierarchy of QECC>

Standard Form of Stabilizer Code

Standard Form

- Matrix Form of Stabilizer Code can be transformed to the so-called **Standard Form** via *Gaussian Elimination* as follows,

$$\mathcal{G} = \left[\begin{array}{c|c} G_X & G_Z \end{array}\right] \xrightarrow{\text{GE}} \mathcal{G} = \left[\begin{array}{ccc|ccc} I & A_1 & A_2 & B & O & C \\ O & O & O & D & I & E \end{array}\right]$$

$$I, B \in \mathrm{GF}(2)^{r\times r} \qquad D \in \mathrm{GF}(2)^{(n-k-r)\times r}$$

$$A_1, O \in \mathrm{GF}(2)^{r\times(n-k-r)} \qquad I \in \mathrm{GF}(2)^{(n-k-r)\times(n-k-r)}$$

$$A_2, C \in \mathrm{GF}(2)^{r\times k} \qquad E \in \mathrm{GF}(2)^{(n-k-r)\times k}$$

Encoded Pauli Operator

- Encoded Pauli operators $\bar{X}$ and $\bar{Z}$ can be found from Standard Form as follows,

$$\mathcal{G} = \left[\begin{array}{ccc|ccc} I & A_1 & A_2 & B & O & C \\ O & O & O & D & I & E \end{array}\right]$$

$$\bar{X} = \left[\begin{array}{ccc|ccc} O & E^T & I & C^T & O & O \end{array}\right]$$
$$\bar{Z} = \left[\begin{array}{ccc|ccc} O & O & O & {A_2}^T & O & I \end{array}\right]$$

- They satisfy the conditions
  - Stabilizer and encoded Pauli X (Z) are mutually commutable.
  - The encoded Pauli-X $\bar{X}_i$ are mutually anti-commuting with $\bar{Z}_i$, but commutable with others $\bar{Z}_j$.

$$\{\bar{X}_i, \bar{Z}_i\} = [\bar{X}_i, \bar{Z}_j] = 0$$

Effective Error Detection

- For a certain stabilizer code, there exist so many different stabilizer generator sets.
- For example, for [[7,1,3]] Steane code,

$$\mathcal{G}^{[[7,1,3]]} = \begin{pmatrix} I & I & I & X & X & X & X \\ I & X & X & I & I & X & X \\ X & I & X & I & X & I & X \\ I & I & I & Z & Z & Z & Z \\ I & Z & Z & I & I & Z & Z \\ Z & I & Z & I & Z & I & Z \end{pmatrix} \quad \mathcal{G}^{[[7,1,3]]'} = \begin{pmatrix} X & I & I & I & X & X & X \\ I & X & I & X & I & X & X \\ I & I & X & X & X & X & I \\ Z & I & Z & Z & I & I & Z \\ I & Z & Z & I & Z & I & Z \\ Z & Z & Z & I & I & Z & I \end{pmatrix}$$

for an error X_1, both stabilizers give the syndromes (+1,+1,+1,+1,+1,-1) and (+1,+1,+1,-1,+1,-1) respectively.

- For practical use, we need a standard stabilizer

Encoding, Decoding, and Correction with Stabilizer Code

Encoding (1/2)

- Encoding by projection onto common eigenvector with an eigenvalue +1
- Note that $(1 + g_i)/2$ is a projector onto the +1 eigenspace of g_i.

$$
\begin{aligned}
|0\rangle_L &= \Big(\sum_{i=1}^{n-k} g_i\Big)|0\cdots 0\rangle \\
&= \frac{1}{2^{n-k}}(1+g_1)(1+g_2)\cdots(1+g_{n-k})|0\cdots 0\rangle
\end{aligned}
$$

- $|0\rangle_L = \cap_{i=1}^{n-k} +1\ eigenspace\ of\ g_i$

Encoding (2/2)

- From Standard Form, it is possible to generate an encoding circuit.

$$G_1 = X \otimes X \otimes X \otimes X \otimes X \otimes X \otimes X \otimes X$$

$$G_2 = Z \otimes Z \otimes Z \otimes Z \otimes Z \otimes Z \otimes Z \otimes Z$$

$$G_3 = X \otimes I \otimes X \otimes I \otimes Z \otimes Y \otimes Z \otimes Y$$

$$G_4 = X \otimes I \otimes Y \otimes Z \otimes X \otimes I \otimes Y \otimes Z$$

$$G_5 = X \otimes Z \otimes I \otimes Y \otimes I \otimes Y \otimes X \otimes Z .$$

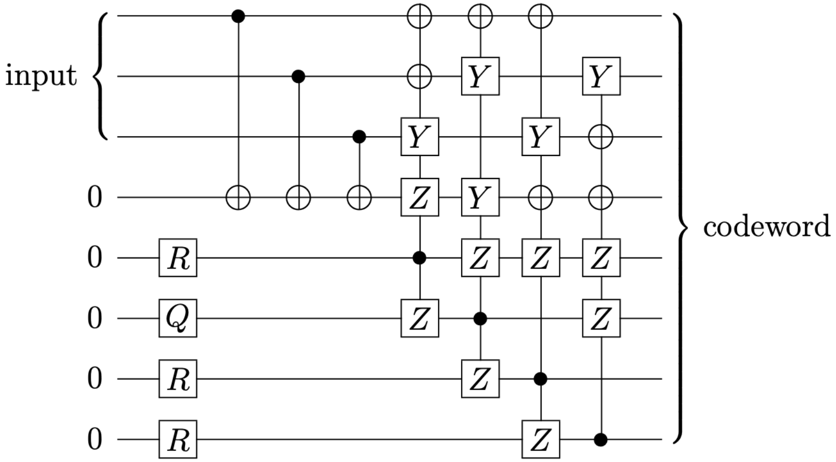
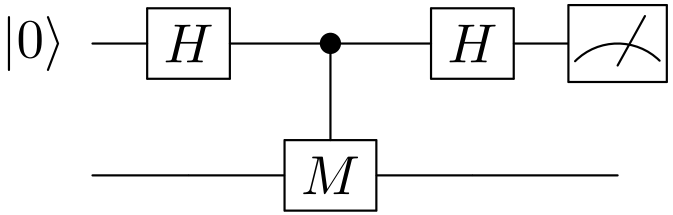


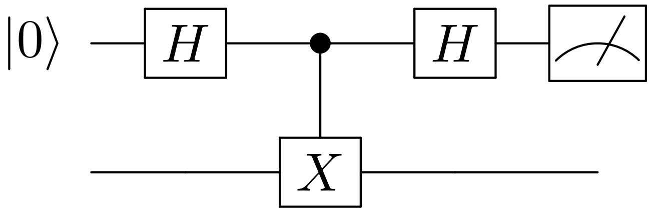
Figure 3

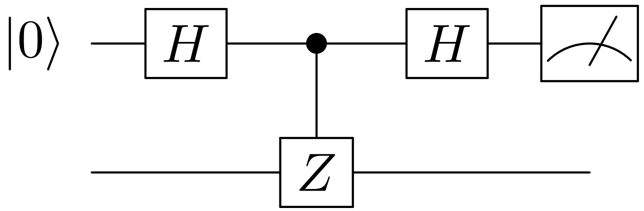
Measure X/Z

- Quantum circuit for measuring a single qubit operator *M* with eigenvalues ± 1



- The following circuits are used to measure X (Z) stabilized 1-qubit state.





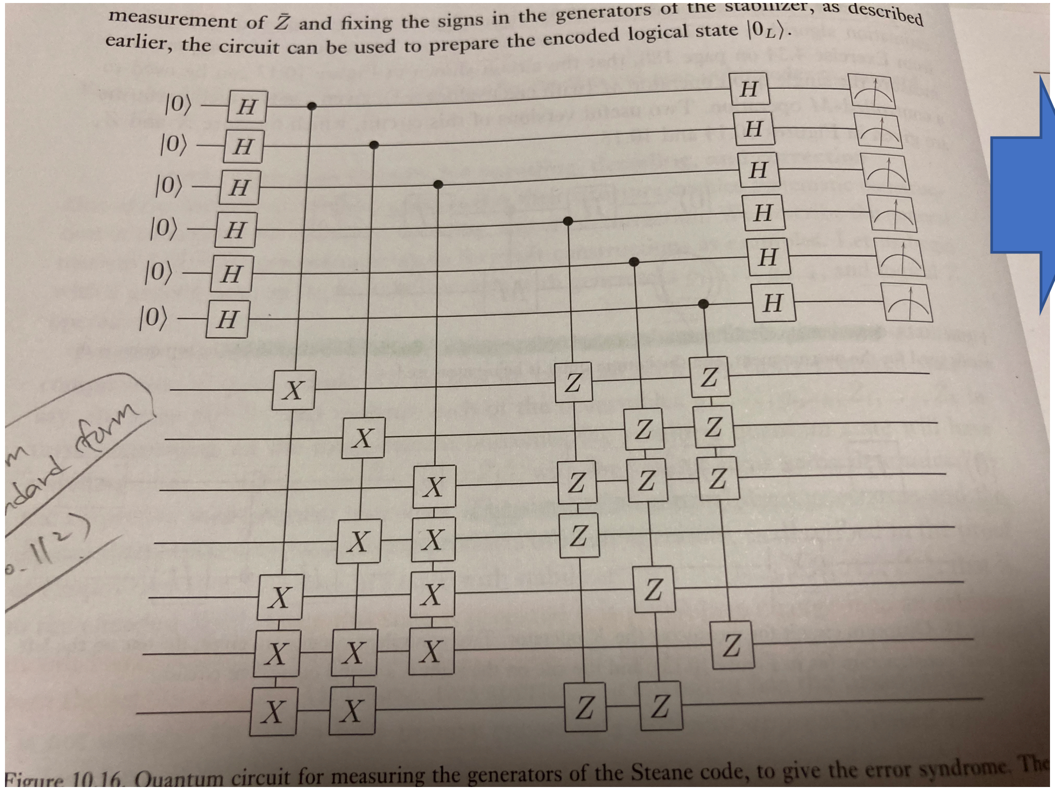
$|+\rangle \longrightarrow 0$

$|-\rangle \longrightarrow 1$

$|0\rangle$ / $|1\rangle$ $\longrightarrow$ 0 / 1 w.p. 0.5

Measure Stabilizer

$$\mathcal{G}^{[[7,1,3]]'} = \begin{pmatrix} X & I & I & I & X & X & X \\ I & X & I & X & I & X & X \\ I & I & X & X & X & X & I \\ Z & I & Z & Z & I & I & Z \\ I & Z & Z & I & Z & I & Z \\ Z & Z & Z & I & I & Z & I \end{pmatrix}$$



Error syndrome

Decoding / Correction

- **(Decoding)** Generally, infer a quantum error from syndrome outcomes.
  - For a fixed sized code, it is possible to build up a ***look up table*** beforehand.
  - For an [[n,k,d]] stabilizer code → table of size 2^{n-k}
  - For example [[3,1,3]] Bit-Flip code,

Z_1Z_2	Z_2Z_3	Error
+1	+1	No Error
+1	-1	X_3
-1	+1	X_1
-1	-1	X_2

- **(Correction)** Apply the inverse of the decoded quantum error.


Thank you