2. Distance Measures : trace distance & fidelity

(Main) Ref. Nielsen & Chuang, Quantum Computation & Quantum Information

① Distance measures for classical info

$\{p_x\}, \{q_x\}$: prob. dist.

$$D(p_x, q_x) := \frac{1}{2}\sum_x |p_x - q_x|$$ total variation distance, L_1 distance, Kolmogorov distance

$$F(p_x, q_x) := \sum_x \sqrt{p_x q_x}$$ fidelity of $\{p_x\}$ and $\{q_x\}$

Prop. $D(p_x, q_x) = \max_{S\subset\{x\}} \left|\sum_{x\in S} p_x - \sum_{x\in S} q_x\right|$

Proof. For $S \subseteq \{x\}$, Let $p(S) = \sum_{x\in S} p_x$, $q(S) = \sum_{x\in S} q_x$. and

$$B := \{x \mid p_x \geq q_x\}.$$

Note $\forall S\subset\{x\}$, $p(S) - q(S) \leq p(S\cap B) - q(S\cap B) \leq p(B) - q(B)$

and $q(S) - p(S) \leq q(B^c) - p(B^c) = p(B) - q(B)$

$p(B^c) = 1 - p(B)$
$q(B^c) = 1 - q(B)$

Thus, $\forall S\subset\{x\}$, $|p(S) - q(S)| \leq p(B) - q(B)$.

$$D(p_x, q_x) = \frac{1}{2}\left(\sum_{x\in B}|p_x - q_x| + \sum_{x\in B^c}|p_x - q_x|\right)$$
$$= \frac{1}{2}\left(p(B) - q(B) + q(B^c) - p(B^c)\right) = p(B) - q(B)$$
$$= \max_{S\subset\{x\}} |p(S) - q(S)| \quad \lrcorner$$

⑥ Distance between two quantum states

§1. trace distance.

ρ, σ = states $\quad D(\rho,\sigma) := \frac{1}{2}\mathrm{tr}|\rho-\sigma|$

trace distance between ρ and σ

$$|A| = \sqrt{A^\dagger A}$$

(e.g.)

① $\rho = \sum_i r_i |i\rangle\langle i|$, $\sigma = \sum_i s_i |i\rangle\langle i|$

$$D(\rho,\sigma) = \frac{1}{2}\,\mathrm{tr}\left|\sum_i (r_i - s_i)|i\rangle\langle i|\right| = D(r_i, s_i)$$

Hermitian

$A^\dagger = A$ $\quad A = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|$ spectral decomp

$\sqrt{A^\dagger A} = \sum_j |\lambda_j| |\phi_j\rangle\langle\phi_j|$

$\therefore \mathrm{tr}|A| = \sum_j |\lambda_j|$

② $\rho = \frac{1}{2}(I_2 + \vec{r}\cdot\vec{\sigma})$, $\sigma = \frac{1}{2}(I_2 + \vec{s}\cdot\vec{\sigma})$ $\quad$ 1-qubit states

$\vec{r} = (r_1, r_2, r_3)$, $\vec{s} = (s_1, s_2, s_3) \in \mathbb{R}^3$

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ $\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$(\vec{r}\cdot\vec{\sigma} = r_1\sigma_x + r_2\sigma_y + r_3\sigma_z)$

Note: $\vec{a} = (a_1, a_2, a_3)$, $\quad \vec{a}\cdot\vec{\sigma} = \begin{pmatrix} a_3 & a_1 - a_2 i \\ a_1 + a_2 i & -a_3 \end{pmatrix}$ is Hermitian

$\mathrm{tr}(\vec{a}\cdot\vec{\sigma}) = 0$, $\det(\vec{a}\cdot\vec{\sigma}) = -|\vec{a}|^2$

So, $\vec{a}\cdot\vec{\sigma}$ has two eigenvalues, $-|\vec{a}|$ and $|\vec{a}|$.

$\therefore \mathrm{tr}|\vec{a}\cdot\vec{\sigma}| = 2|\vec{a}|$

$$D(\rho,\sigma) = \frac{1}{2}\mathrm{tr}|\rho-\sigma| = \frac{1}{2}\mathrm{tr}\left|\frac{1}{2}(\vec{r}-\vec{s})\cdot\vec{\sigma}\right| = \frac{1}{4}\mathrm{tr}|(\vec{r}-\vec{s})\cdot\vec{\sigma}|$$
$$= \frac{1}{4}\times 2|\vec{r}-\vec{s}| = \frac{1}{2}|\vec{r}-\vec{s}|$$

Thm 1 $\{E_m\}$: POVM $\quad E_m \geq 0 \quad \sum_m E_m = I$

ρ, σ = states $\quad \rho$: state $\quad p(m) = \mathrm{tr}(E_m\rho)$

$$p_m := \mathrm{tr}(E_m\rho),\ q_m = \mathrm{tr}(E_m\sigma) \Longrightarrow D(\rho,\sigma) = \max_{\{E_m\}} D(p_m, q_m)$$

proof. By def, $D(p_m, q_m) = \frac{1}{2}\sum_m |p_m - q_m| = \frac{1}{2}\sum_m |\mathrm{tr}(E_m(\rho-\sigma))|$.

Since $\rho-\sigma$ is Hermitian, $\exists\, Q, S \geq 0$ s.t. $\mathrm{tr}(QS) = 0$, $\rho - \sigma = Q - S$ & $|\rho-\sigma| = Q+S$

Thus, $D(p_m, q_m) = \frac{1}{2}\sum_m |\mathrm{tr}(E_m(Q-S))| \leq \frac{1}{2}\sum_m \mathrm{tr}(E_m(Q+S)) = \frac{1}{2}\sum_m \mathrm{tr}(E_m|\rho-\sigma|)$
$= \frac{1}{2}\mathrm{tr}|\rho-\sigma| = D(\rho,\sigma)$

Let P be the projector onto $\mathrm{supp}(Q)$. $(PQ = Q$ and $PS = 0)$

Then $\{P, I-P\}$ is a POVM. $\quad D(p_i, q_i) = \frac{1}{2}|\mathrm{tr}\,P(\rho-\sigma) + \mathrm{tr}((I-P)(\rho-\sigma))| = \frac{1}{2}|\mathrm{tr}\,P(Q-S) + \mathrm{tr}((I-P)$

$p_1 = \mathrm{tr}(P\rho)$, $p_2 = \mathrm{tr}((I-P)\rho)$

$q_1 = \mathrm{tr}(P\sigma)$, $q_2 = \mathrm{tr}((I-P)\sigma)$

$= \frac{1}{2}|\mathrm{tr}\,Q - \mathrm{tr}\,S| = \frac{1}{2}\mathrm{tr}(Q+S) = \frac{1}{2}\mathrm{tr}|\rho-\sigma| = D(\rho,\sigma)$

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In the proof of Thm 1, $D(\rho,\sigma) = \frac{1}{2}\mathrm{tr}(Q+S)$,
where $\rho - \sigma = Q - S$, $|\rho-\sigma| = Q+S$.

Since $0 = \mathrm{tr}(\rho-\sigma) = \mathrm{tr}(Q-S)$, $\mathrm{tr}(Q) = \mathrm{tr}(S)$.

$\therefore D(\rho,\sigma) = \frac{1}{2}\mathrm{tr}(Q+S) = \mathrm{tr}(Q) = \mathrm{tr}(P(Q-S))$
$= \mathrm{tr}(P(\rho-\sigma))$, where P is the projector onto $\mathrm{supp}(Q)$.

Prop. $D(\rho,\sigma) = \max\limits_{P:\text{ projector}} \mathrm{tr}(P(\rho-\sigma))$

Thm 2 $\mathcal{E} = (\mathrm{CP})\mathrm{TP}$. ρ,σ : states

$\Rightarrow D(\mathcal{E}(\rho),\mathcal{E}(\sigma)) \le D(\rho,\sigma)$

quantum data processing ineq. w.r.t. trace distance
($\mathcal{E}$: contractive)

proof Note: $\rho - \sigma = Q - S$ for some $Q, S \geq 0$
satisfying $\mathrm{tr}(QS) = 0$, $|\rho - \sigma| = Q+S$.

$D(\rho,\sigma) = \frac{1}{2}\mathrm{tr}|\rho-\sigma| = \frac{1}{2}\mathrm{tr}(Q+S) = \mathrm{tr}(Q)$ ($\because 0 = \mathrm{tr}(\rho-\sigma) = \mathrm{tr}(Q-S)$)

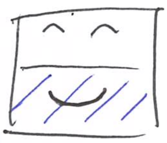
$\underset{\uparrow \mathrm{TP}}{=} \mathrm{tr}(\mathcal{E}(Q)) \geq \mathrm{tr}(P(\mathcal{E}(Q)))$, where P is the projector s.t. $D(\mathcal{E}(\rho),\mathcal{E}(\sigma)) = \mathrm{tr}(P(\mathcal{E}(\rho)-\mathcal{E}(\sigma)))$

$\geq \mathrm{tr}(P(\mathcal{E}(Q)-\mathcal{E}(S)))$
$= \mathrm{tr}(P(\mathcal{E}(Q-S))) = \mathrm{tr}(P\mathcal{E}(\rho-\sigma))$
$= \mathrm{tr}(P(\mathcal{E}(\rho)-\mathcal{E}(\sigma))) = D(\mathcal{E}(\rho),\mathcal{E}(\sigma))$. $\lrcorner$

partial trace is trace-preserving

(If $\mathrm{tr}(\rho_{AB}) = \mathrm{tr}(\sigma_{AB})$ then
$\mathrm{tr}(\mathrm{tr}_A(\rho_{AB})) = \mathrm{tr}(\mathrm{tr}_A(\sigma_{AB}))$
$\quad\ \ \shortparallel \qquad\qquad\qquad \shortparallel$
$\mathrm{tr}(\rho_{AB}) \qquad\quad \mathrm{tr}(\sigma_{AB})$)

Cor. $D(\rho_A,\sigma_A) \le D(\rho_{AB},\sigma_{AB})$




$\frac{1}{2}I = \mathrm{tr}_A(|\Phi^+\rangle\langle\Phi^+|_{AB}) = \mathrm{tr}(|\Phi^-\rangle\langle\Phi^-|_{AB})$

$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B$

§2. Fidelity

$F(p_m, q_m) = \sum_m \sqrt{p_m q_m}$ classical.

ρ, σ : states $F(\rho,\sigma) := \mathrm{tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}$ fidelity of ρ and σ

(e.g) ① $\rho = \sum_i r_i |i\rangle\langle i|$, $\sigma = \sum_i s_i |i\rangle\langle i|$ $(r_i, s_i \geq 0)$

$$F(\rho,\sigma) = \mathrm{tr}\sqrt{\left(\sum_i \sqrt{r_i}|i\rangle\langle i|\right)\sum_i s_i|i\rangle\langle i|\left(\sum_i\sqrt{r_i}|i\rangle\langle i|\right)} = \mathrm{tr}\sqrt{\sum_i r_i s_i |i\rangle\langle i|}$$

$$= \mathrm{tr}\left(\sum_i \sqrt{r_i s_i}\,|i\rangle\langle i|\right) = \sum_i \sqrt{r_i s_i} = F(r_i, s_i)$$

② $$F(|\psi\rangle\langle\psi|, \rho) = \mathrm{tr}\sqrt{|\psi\rangle\underbrace{\langle\psi|\rho|\psi\rangle}_{\text{scalar}}\langle\psi|} = \mathrm{tr}\sqrt{\langle\psi|\rho|\psi\rangle\cdot|\psi\rangle\langle\psi|}$$

$$= \mathrm{tr}\left(\sqrt{\langle\psi|\rho|\psi\rangle}\,|\psi\rangle\langle\psi|\right) = \sqrt{\langle\psi|\rho|\psi\rangle}$$

③ $$F(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) = \sqrt{\langle\psi|\phi\rangle\langle\phi|\psi\rangle} = |\langle\psi|\phi\rangle| .$$

Thm (Uhlmann)

ρ, σ : states of a system Q.

$$\Rightarrow F(\rho,\sigma) = \max_{|\psi\rangle, |\phi\rangle} |\langle\psi|\phi\rangle| \quad \begin{pmatrix} |\psi\rangle_{RQ} : \text{purification of } \rho \\ |\phi\rangle_{RQ} : \quad \text{〃} \quad \text{of } \sigma \end{pmatrix}$$

$0 \leq F(p_m, q_m) = \sum_m \sqrt{p_m q_m} \leq 1 = F(p_m, p_m)$

|inner product|

$F(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) = |\langle\psi|\phi\rangle|$

$F(\rho,\sigma) = ?$ Uhlmann's definition is natural.

Lemma. A: matrix , U: unitary $\Rightarrow |\mathrm{tr}(AU)| \leq \mathrm{tr}|A|$

("=" $\Leftrightarrow U = V^\dagger$, where $A = |A|V$ polar decomp)

proof. $|\mathrm{tr}(AU)| \overset{\text{polar decomp.}}{=} |\mathrm{tr}(|A|VU)| = |\mathrm{tr}(\sqrt{|A|}\sqrt{|A|}VU)|$

$\leq \sqrt{\mathrm{tr}(\sqrt{|A|}\sqrt{|A|})}\sqrt{\mathrm{tr}(U^\dagger V^\dagger\sqrt{|A|}\sqrt{|A|}VU)} = \mathrm{tr}|A|$

Cauchy-Schwarz ineq. w.r.t Hilbert-Schmidt inner product

"=" $\Leftrightarrow VU = I \Leftrightarrow U = V^\dagger$

If $U = V^\dagger$ then $|\mathrm{tr}(AU)| = |\mathrm{tr}(|A|VU)| = \mathrm{tr}|A|$.

Prop. $|\Phi\rangle = \sum_i |i\rangle \otimes |i\rangle$

$$\langle \Phi | A \otimes B | \Phi \rangle = \mathrm{tr}(A^T B)$$

⌜proof $\langle \Phi | A \otimes B | \Phi \rangle = \sum_{i,j} (\langle i | \otimes \langle i |)(A \otimes B)(|j\rangle \otimes |j\rangle)$

$= \sum_{i,j} \langle i | A | j \rangle \langle i | B | j \rangle = \sum_{i,j} \langle j | A^T | i \rangle \langle i | B | j \rangle$

$= \sum_j \langle j | A^T B | j \rangle = \mathrm{tr}(A^T B)$ ⌟

$\left(A \otimes B | \Phi \rangle = (I \otimes B A^T) | \Phi \rangle \right)$ option.

Thm (Uhlmann)

ρ, σ = states of Q. $\Longrightarrow$ $F(\rho, \sigma) = \max_{|\psi\rangle_{RQ}, |\phi\rangle_{RQ}} |\langle \psi | \phi \rangle|$

purifications of ρ & σ, resp.

⌜proof $\rho = \mathrm{tr}_R(|\psi\rangle\langle\psi|_{RQ})$ and $\sigma = \mathrm{tr}_R(|\phi\rangle\langle\phi|_{RQ})$.

$|\Phi\rangle_{RQ} := \sum_i |i\rangle_R \otimes |i\rangle_Q$

If $\rho = \sum_i \lambda_i |i\rangle\langle i|$, $(I \otimes \sqrt{\rho}) |\Phi\rangle_{RQ} = \sum_i \sqrt{\lambda_i} |i\rangle_R \otimes |i\rangle_Q$

By Schmidt decomp., $|\psi\rangle = (U_R \otimes \sqrt{\rho}\, U_Q) |\Phi\rangle_{RQ}$ for some unitary U_R, U_Q, V_R, V_Q

$|\phi\rangle = (V_R \otimes \sqrt{\sigma}\, V_Q) |\Phi\rangle_{RQ}$

$|\langle \psi | \phi \rangle| = |\langle \phi | \psi \rangle| = |\langle \Phi | V_R^\dagger U_R \otimes V_Q^\dagger \sqrt{\sigma} \sqrt{\rho}\, U_Q | \Phi_{RQ} \rangle|$

$\overset{\text{Prop.}}{=} |\mathrm{tr}(U_R^T V_R^* V_Q^\dagger \sqrt{\sigma}\sqrt{\rho}\, U_Q)| = |\mathrm{tr}(\sqrt{\sigma}\sqrt{\rho}\, \underbrace{U_Q U_R^T V_R^* V_Q^\dagger}_{\text{unitary}})|$

$\overset{\text{Lemma}}{\le} \mathrm{tr}|\sqrt{\sigma}\sqrt{\rho}| = \mathrm{tr}\sqrt{(\sqrt{\sigma}\sqrt{\rho})^\dagger (\sqrt{\sigma}\sqrt{\rho})} = \mathrm{tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}} = F(\rho, \sigma)$.

If $(U_Q U_R^T V_R^* V_Q^\dagger)^\dagger = V$, where $\sqrt{\sigma}\sqrt{\rho} = |\sqrt{\sigma}\sqrt{\rho}| V$ is polar decomp of $\sqrt{\sigma}\sqrt{\rho}$,

then $F(\rho, \sigma) = |\langle \psi | \phi \rangle|$. ⌟

- Properties of fidelity

(Monotone) $\mathcal{E}$ = CPTP. ρ, σ = states

$$F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho, \sigma)$$

(concave)

$$F\left(\sum_i p_i \rho_i, \sigma\right) \geq \sum_i p_i F(\rho_i, \sigma)$$

(Jointly concave)

$$F\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i\right) \geq \sum_i p_i F(\rho_i, \sigma_i)$$

(strongly concave)

$$F\left(\sum_i p_i \rho_i, \sum_i q_i \sigma_i\right) \geq \sum_i \sqrt{p_i q_i}\, F(\rho_i, \sigma_i)$$

<u>prop.</u> $F(\rho, \sigma) = \min_{\{E_m\}\ \text{POVM}} F(p_m, q_m)$, where $p_m = tr(E_m \rho)$, $q_m = tr(E_m \sigma)$

$|\psi_i\rangle$: purification of ρ_i $\quad F(\rho_i, \sigma_i) = |\langle \psi_i | \phi_i \rangle|$

$|\phi_i\rangle$: " of σ_i

$|\psi\rangle = \sum_i \sqrt{p_i}\, |\psi_i\rangle_{RQ} \otimes |i\rangle_{R'}$ $\quad$ purifications of $\sum_i p_i \rho_i$

$|\phi\rangle = \sum_i \sqrt{q_i}\, |\phi_i\rangle_{RQ} \otimes |i\rangle_{R'}$ $\quad \sum_i q_i \sigma_i$

$F\left(\sum_i p_i \rho_i, \sum_i q_i \sigma_i\right) \geq |\langle \psi | \phi \rangle|$

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§3. Relationships between fidelity & trace distance.

$F(\rho, \sigma) = F(p_m, q_m)$ for some $\{E_m\}$ POVM

$= \sum_m \sqrt{p_m q_m}$

$$2(1 - F(\rho,\sigma)) = 2\left(1 - \sum_m \sqrt{p_m q_m}\right) = \sum_m \left(\sqrt{p_m} - \sqrt{q_m}\right)^2 \leq \sum_m \left|\sqrt{p_m} - \sqrt{q_m}\right| \left|\sqrt{p_m} + \sqrt{q_m}\right|$$

$$= \sum_m |p_m - q_m|$$

$$= 2 D(p_m, q_m)$$

$$\leq 2 D(\rho, \sigma)$$

$\therefore$ $\underline{1 - F(\rho, \sigma) \leq D(\rho, \sigma)}$

By D.P.I of trace distance, $D(\rho,\sigma) \leq D(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|)$

$= \frac{1}{2} tr \left| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \right|$

$|\psi\rangle$: purification of ρ

$|\phi\rangle$: " of σ

$F(\rho, \sigma) = |\langle\psi|\phi\rangle|$

$|\phi\rangle = \alpha|\psi\rangle + \beta|\psi^\perp\rangle$

$|\phi\rangle\langle\phi| = |\alpha|^2 |\psi\rangle\langle\psi| + \alpha\beta^* |\psi\rangle\langle\psi^\perp| + \alpha^*\beta |\psi^\perp\rangle\langle\psi| + |\beta|^2 |\psi^\perp\rangle\langle\psi^\perp|$

$$= \frac{1}{2} tr \left| \underbrace{\begin{pmatrix} 1 - |\alpha|^2 & -\alpha\beta^* \\ -\alpha^*\beta & -|\beta|^2 \end{pmatrix}}_{G} \right|$$

$tr(G) = 0$, $\det(G) = -(1 - |\alpha|^2)|\beta|^2 - |\alpha|^2|\beta|^2$

$= -|\beta|^2 = -(1 - |\langle\psi|\phi\rangle|^2)$

eigenvalues, $-\sqrt{1 - |\langle\psi|\phi\rangle|^2}$, $\sqrt{1 - |\langle\psi|\phi\rangle|^2}$

$\therefore D(\rho, \sigma) \leq \frac{1}{2} \cdot 2\sqrt{1 - |\langle\psi|\phi\rangle|^2}$

$= \sqrt{1 - F(\rho,\sigma)^2}$

$\therefore$ $1 - F(\rho,\sigma) \leq D(\rho,\sigma) \leq \sqrt{1 - F(\rho,\sigma)^2}$

§4. Applications

- Teleportation fidelity

$$F(\Lambda_\rho) = \int \underbrace{\langle\psi|\Lambda_{\rho_{AB}}(\psi)|\psi\rangle}_{= F(|\psi\rangle\langle\psi|,\ \Lambda_{\rho_{AB}}(\psi))^2} d\psi$$

$$|\psi\rangle \longmapsto \Lambda_{\rho_{AB}}(\psi)$$

$f(\rho) = \max_{|e\rangle : \text{m.e}} \langle e|\rho|e\rangle$ fully entangled fraction

$F(\Lambda_\rho) = (d f(\rho) + 1)/(d+1)$

- Entanglement fidelity

$$F(\rho, \mathcal{E}) := \langle\psi|(I_R \otimes \mathcal{E})(|\psi\rangle\langle\psi|_{RQ})|\psi\rangle_{RQ} = F(|\psi\rangle\langle\psi|_{RQ}, (I_R\otimes\mathcal{E})(|\psi\rangle\langle\psi|_{RQ}))$$

ρ: state of Q; $\mathcal{E}$: CPTP on $\mathcal{H}_Q$; RQ

$|\psi\rangle_{RQ}$: purification of ρ_Q

How well is entanglement between R and Q preserved by $\mathcal{E}$

- Operational interpretation of trace distance

ρ_0, ρ_1 : states $\qquad \begin{cases} \rho_0 & \text{with prob } \frac{1}{2} \\ \rho_1 & \text{〃} \end{cases}$

$E = \{E_0, E_1\}$: POVM $\quad E_0 \to \rho_0,\ E_1 \to \rho_1 \quad E_0 + E_1 = I$

$P_{succ}(E)$ = correctly guessing prob. with E.

$$Q: \quad P_{succ} = \max_{E=\{E_0,E_1\}} P_{succ}(E) = ?$$

$$P_{succ}(E) = \frac{1}{2}\left(\mathrm{tr}(E_0\rho_0) + \mathrm{tr}(E_1\rho_1)\right) = \frac{1}{2}\left(\mathrm{tr}(E_0\rho_0) + 1 - \mathrm{tr}(E_0\rho_1)\right)$$

$$= \frac{1}{2}\left(1 + \mathrm{tr}(E_0(\rho_0 - \rho_1))\right) \le \frac{1}{2}\left(1 + \max_{E_0 \ge 0} \mathrm{tr}(E_0(\rho_0-\rho_1))\right)$$

$$= \frac{1}{2}\left(1 + \underline{D(\rho_0, \rho_1)}\right)$$

$E_0 = \sum_i \lambda_i P_i$ spectral decomp.

$0 \le \lambda_i \le 1$ ($\because E_0 + E_1 = I$)

$P := \sum_i P_i$ projection

END.